

Use of singular classical solutions for calculation of multiparticle cross sections in field theory.

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Abstract

A method of reducing the problem of the calculation of tree multiparticle cross sections in φ^4 theory to the solution of a singular classical Euclidean boundary value problem is introduced. The solutions are obtained numerically in terms of the decomposition in spherical harmonics, and the corresponding estimates of the tree cross sections at arbitrary energies are found. Numerical analysis agrees with analytical results obtained earlier in the limiting cases of large and small energies.

1 Introduction

The main theoretical tool in quantum field theory is presently the perturbation theory. Perturbative calculations provided the majority of the experimentally checked results. Therefore, limits of applicability of perturbative calculations are of considerable interest. On the one hand, there exist processes related to complex vacuum structure in gauge theories and nontrivial classical solutions to field equations which cannot be described by perturbation theory. On the other hand, even in the topologically trivial sector at relatively low energies, processes which are described poorly by perturbation theory also exist.

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Perturbative calculations provide reliable results only in weakly coupled models where the expansion parameter—dimensionless coupling constant—is much smaller than unity. But even in such theories situations are possible in which other competing small (or large) parameters exist. A typical example is a process with large number n of particles in the final state (n being of the order of the inverse coupling constant λ^{-1}).

In conventional perturbation theory even above the topologically trivial vacuum, the naive estimate of the amplitude gives the factorial dependence $n!$ on the multiplicity of the final state. This enhancement can in principle overcome the suppression due to the powers of the coupling constant. At the tree level, it is possible to find an exact expression for the amplitude of creation of n real particles by one virtual particle in the theory with lagrangian density

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\varphi)^2 - \frac{1}{2}\varphi^2 - \frac{\lambda}{4}\varphi^4 \quad (1)$$

(the mass is set equal to one) in special kinematics, namely, when all particles have zero spatial momenta [1],

$$A_{1\rightarrow n}^{\text{tree}} = n! \left(\frac{\lambda}{8} \right)^{\frac{n-1}{2}} . \quad (2)$$

This result points towards complete breakdown of the usual perturbative calculations at $n \gtrsim \lambda^{-1}$ because it contradicts unitarity of the theory.

Thus, some non-perturbative method is required for the calculation of these cross sections. The limit we are interested in is

$$\lambda \rightarrow 0 , \quad \lambda n = \text{fixed} , \quad \varepsilon = \text{fixed} , \quad (3)$$

where $\varepsilon = (E - n)/n$ is the average kinetic energy of the outgoing particles in the centre of mass frame. Existing perturbative calculations [2, 3] strongly suggest that in this limit the total cross section has the exponential form,

$$\sigma_{1\rightarrow n} \sim \exp \left(\frac{1}{\lambda} F(\lambda n, \varepsilon) \right) . \quad (4)$$

This form implies the semiclassical calculability of the cross sections. A method to obtain the exponent $F(\lambda n, \varepsilon)$ in all loops was formulated in ref. [4], which required solution of a certain classical boundary problem in complex time. At small λn one needs only to solve purely Euclidean equations

with special boundary conditions. In conventional perturbative approach this limit corresponds to the contribution of tree graphs, that gives the following dependence on λ :

$$F_{\text{tree}}(\lambda n, \varepsilon) = \lambda n \ln \left(\frac{\lambda n}{16} \right) - \lambda n + \lambda n f(\varepsilon) . \quad (5)$$

Let us note that in the domain of its applicability, i. e., at $\lambda n \ll 1$, this dependence means the exponential suppression of the cross section, at least if $f(\varepsilon)$ does not become infinite. But as λn increases, the function $F_{\text{tree}}(\lambda n, \varepsilon)$ becomes positive and this suppression disappears. Thus, in the latter case one has to take into account loop corrections to $F(\lambda n, \varepsilon)$ which are of the order $(\lambda n)^2$ and higher [3].

Making use of the technique developed in refs. [4, 5] it is possible, at least in principle, to find the only unknown function $f(\varepsilon)$ in (5).

Even in the simplest case of small λn the calculation of the exponent F_{tree} at all energies ε is rather complicated (the method of the calculation is described in section 2). The corresponding classical solution has singularities on a three-dimensional surface in four-dimensional Euclidean space (in the case of four dimensional theory). This surface depends on ε and is determined in the course of the calculation. The Rayleigh–Ritz variational procedure enables one to obtain the lower bound on F_{tree} . Computational procedure and its numerical realization are described in sections 3 and 4. The purpose of this paper is to explore the possibilities for the actual calculation of the tree exponent (5). In section 5 our results are compared to known analytical results in the limiting cases.

2 Singular solutions and tree cross sections. General formalism

Let us describe the technique to obtain the exponent for tree cross sections in the limit (3) [4, 5, 6]. We consider the process of the decay of one virtual particle of energy E and momentum $\mathbf{P} = 0$ into n real particles in the model with the lagrangian density (1). Let us write the matrix element $\langle \beta | S \varphi | 0 \rangle$ in the coherent state representation [7, 8, 9], in $(d+1)$ –dimensional Minkowskian

space-time ($\langle\beta|$ is a coherent state)

$$\langle\beta|S\varphi|0\rangle = \lim_{\substack{T_i \rightarrow -\infty \\ T_f \rightarrow +\infty}} \int D\varphi D\varphi_i D\varphi_f \varphi(E, \mathbf{P}) e^{B_i(\varphi_i) + B_f(\beta^*, \varphi_f) + i \int \mathcal{L} d^{d+1}x}, \quad (6)$$

where

$$\begin{aligned} \varphi(E, \mathbf{P}) &= \int dt d^d x \varphi(t, \mathbf{x}) e^{-iEt + i\mathbf{P}\mathbf{x}}, \quad \varphi_{\mathbf{k}}(t) = \int \frac{d^d x}{(2\pi)^{d/2}} \varphi(t, \mathbf{x}) e^{-i\mathbf{k}\mathbf{x}}, \\ \varphi_i(\mathbf{k}) &= \varphi_{\mathbf{k}}(T_i), \quad \varphi_f(\mathbf{k}) = \varphi_{\mathbf{k}}(T_f), \end{aligned}$$

and the boundary terms are

$$\begin{aligned} B_i(\varphi_i) &= -\frac{1}{2} \int d^d k \omega_{\mathbf{k}} \varphi_i(\mathbf{k}) \varphi_i(-\mathbf{k}), \\ B_f(\beta^*, \varphi_f) &= -\frac{1}{2} \int d^d k \omega_{\mathbf{k}} \varphi_f(\mathbf{k}) \varphi_f(-\mathbf{k}) \\ &\quad -\frac{1}{2} \int d^d k \beta_{\mathbf{k}}^* \beta_{-\mathbf{k}}^* e^{2i\omega_{\mathbf{k}} T_f} + \int d^d k \sqrt{2\omega_{\mathbf{k}}} e^{i\omega_{\mathbf{k}} T_f} \beta_{\mathbf{k}}^* \varphi_f(-\mathbf{k}), \end{aligned}$$

where $\omega_{\mathbf{k}} = \sqrt{1 + \mathbf{k}^2}$. At the tree level the integral (6) is determined by the value of integrand taken at the saddle point. The extremum conditions for the exponent are the classical field equation

$$\partial_\mu^2 \varphi + \varphi + \lambda \varphi^3 = 0 \quad (7)$$

and the following boundary conditions:

$$\varphi_{\mathbf{k}}(t) \xrightarrow{t \rightarrow -\infty} a_{\mathbf{k}} e^{i\omega_{\mathbf{k}} t}, \quad \varphi_{\mathbf{k}}(t) \xrightarrow{t \rightarrow +\infty} \frac{\beta_{\mathbf{k}}^*}{\sqrt{2\omega_{\mathbf{k}}}} e^{i\omega_{\mathbf{k}} t} + c_{\mathbf{k}} e^{-i\omega_{\mathbf{k}} t}, \quad (8)$$

where $a_{\mathbf{k}}$ and $c_{\mathbf{k}}$ are arbitrary. At $t \rightarrow -\infty$ the solution $\varphi_c(\beta^*, t, \mathbf{x})$ has only positive frequency part. Energy conservation then implies that at $t \rightarrow +\infty$ the solution should not contain negative frequency parts either, i. e., $c_{\mathbf{k}} = 0$, and the exponent in eq. (6) is zero. Thus, the matrix element in the coherent state representation has the form of the Fourier component of the saddle point solution,

$$A_E(\beta^*) \equiv \langle\beta|S\varphi|0\rangle_{\text{tree}} = \int dt d^d x \varphi_c(\beta, t, \mathbf{x}) e^{-iEt + i\mathbf{P}\mathbf{x}}. \quad (9)$$

As follows from the coherent state formalism, the tree amplitude of the process $1 \rightarrow n$ can be obtained from the matrix element (9) in the following way,

$$A_{1 \rightarrow n}(\mathbf{k}_1, \dots, \mathbf{k}_n) = \left. \frac{\partial^n A_E(\beta^*)}{\partial \beta_{\mathbf{k}_1}^* \dots \partial \beta_{\mathbf{k}_n}^*} \right|_{\beta^*=0} . \quad (10)$$

To find the n -particle cross section, let us introduce the generating function

$$\Sigma(\xi, E) = \frac{1}{Z} \int D\beta D\beta^* \exp \left\{ - \int d^d k \beta_{\mathbf{k}} \beta_{\mathbf{k}}^* \right\} A_E(\sqrt{\xi} \beta^*) \bar{A}_E(\sqrt{\xi} \beta) , \quad (11)$$

where Z is the normalization factor. The total cross section is then given by the following formula,

$$\sigma_{1 \rightarrow n}^{\text{tree}}(E, n) = \frac{1}{n!} \frac{\partial^n}{\partial \xi^n} \Sigma(\xi, E) \Big|_{\xi=0} .$$

One can check this relation by differentiating the right hand side of eq. (11) and using eq. (10) [5, 10].

Making use of the Cauchy formula one can rewrite the expression for $\sigma_{1 \rightarrow n}^{\text{tree}}$ in the following form

$$\sigma_{1 \rightarrow n}^{\text{tree}} = \frac{1}{Z} \oint \frac{d\xi}{\xi^{n+1}} \int D\beta D\beta^* \exp \left\{ - \frac{1}{\xi} \int d^d k \beta_{\mathbf{k}}^* \beta_{\mathbf{k}} \right\} A_E(\beta^*) \bar{A}_E(\beta) . \quad (12)$$

This integral can be calculated again in the saddle point approximation, after taking care of zero modes corresponding to time translations and possible exponentially large factors in $A_E(\beta^*)$. To get rid of both of them, let us introduce the following variables,

$$\beta_{\mathbf{k}}^* = b_{\mathbf{k}}^* e^{i\omega_{\mathbf{k}} t_0 - i\mathbf{k} \cdot \mathbf{x}_0} .$$

In terms of these variables we have

$$\begin{aligned} \varphi_c(\beta^*, t, \mathbf{x}) &= \varphi(b^*, t + t_0, \mathbf{x} + \mathbf{x}_0) , \\ A_E(\beta^*) &= A_E(b^*) e^{iEt_0 - i\mathbf{P} \cdot \mathbf{x}_0} . \end{aligned}$$

Here t_0, \mathbf{x}_0 are collective coordinates and $b_{\mathbf{k}}$ are new integration variables, obeying some constraint that fixes translational invariance. The form of

this constraint will be determined later. In terms of new variables, eq. (12) becomes

$$\begin{aligned} \sigma_{1 \rightarrow n}^{\text{tree}} = & \frac{1}{Z} \oint \frac{d\xi}{\xi} \int DbDb^* dx_0 dx'_0 J A_E(b^*) \bar{A}_E(b) \times \\ & \times \exp \left[iE(t_0 - t'_0) - i\mathbf{P}(\mathbf{x}_0 - \mathbf{x}'_0) \right. \\ & \left. - \frac{1}{\xi} \int d^d k b_{\mathbf{k}}^* b_{\mathbf{k}} e^{i\omega_{\mathbf{k}}(t_0 - t'_0) - i\mathbf{k}(\mathbf{x}_0 - \mathbf{x}'_0)} - n \ln \xi \right], \end{aligned} \quad (13)$$

where J contains δ -function of the constraint on $b_{\mathbf{k}}$ and the corresponding Faddeev–Popov; the later that does not make exponential contribution and will not be considered in what follows. The integration over $(x_0 + x'_0)$ gives the volume factor canceling out with Z . If there are no more exponentially large factors in A_E , we can use the saddle point approximation (the saddle point of the variable $(\mathbf{x}_0 - \mathbf{x}'_0)$ is equal to zero because we work in the centre of mass frame $\mathbf{P} = 0$; we will not write this variable later on):

$$\sigma^{\text{tree}}(E, n) \propto e^{W_{\text{tree}}^{\text{extr}}}, \quad (14)$$

where $W_{\text{tree}}^{\text{extr}}$ is the extremum value of the functional

$$W_{\text{tree}}(T, \theta, b_{\mathbf{k}}, b_{\mathbf{k}}^*) = ET - n\theta - e^{-\theta} \int d^d k b_{\mathbf{k}}^* b_{\mathbf{k}} e^{\omega_{\mathbf{k}} T} \quad (15)$$

over $T = i(t_0 - t'_0)$, $\theta = \ln \xi$, $b_{\mathbf{k}}$ and $b_{\mathbf{k}}^*$.

Let us now determine the constraint on $b_{\mathbf{k}}$. It should break the translational invariance. We have already mentioned that we need the condition on $b_{\mathbf{k}}$ to get rid of the exponential factors in $A_E(b_{\mathbf{k}})$. Let us continue analytically the solution of eq. (7) to the Euclidean time. Then the boundary condition (8) and the absence of negative frequency parts, $c_{\mathbf{k}} = 0$, implies that the solution should decay at $\text{Im } t = \tau \rightarrow +\infty$, where τ is the Euclidean time. We are not interested in instanton effects, i. e., we do not consider classical solutions regular in Euclidean space (in $\lambda\varphi^4$ theory there are no such solutions if $\lambda > 0$). Then, φ_c should be singular somewhere in Euclidean space-time. Generally, φ_c is singular on some surface $\tau = \tau_s(\mathbf{x})$, where $\tau_s(\mathbf{x}) < 0$ for solutions smooth on the real time axis. The behaviour of the integral (9) is determined by singularities of the function φ_c , i. e., it is proportional to $\exp(E\tau_m + i\mathbf{P}\mathbf{x}_m)$, where τ_m and \mathbf{x}_m are coordinates of the singularity closest to the real axis ($\tau_m < 0$). Thus, to get rid of exponential factors in $A_E(\beta^*)$

we need $\tau_m \rightarrow 0$, $\mathbf{x}_m = 0$. This means, in other words, that we require the singularity surface in Euclidean space-time to touch¹ the plane $\tau = 0$ at the point $\mathbf{x} = 0$, i. e., $\tau_s(\mathbf{x} = 0) = 0$; $\tau_s(\mathbf{x}) < 0$ at $\mathbf{x} \neq 0$. This condition simultaneously fixes the translational invariance (in complex time), so it is indeed a constraint fixing zero modes.

So, the problem of finding the tree cross sections at any E and n can be formulated in Euclidean space-time and consists of the following steps:

- Find $O(d)$ -symmetric solutions $\varphi(\tau, \mathbf{x})$ of the Euclidean field equations

$$\partial^2 \varphi - \varphi - \lambda \varphi^3 = 0, \quad (16)$$

which is singular on the surface $\tau_s(\mathbf{x}) \leq 0$, $\tau_s(0) = 0$ and has the following asymptotics at $\tau \rightarrow \infty$:

$$\int \frac{d^d x}{(2\pi)^{d/2}} \varphi(\tau, \mathbf{x}) e^{-i\mathbf{k}\mathbf{x}} = \frac{b_{\mathbf{k}}^*}{\sqrt{2\omega_{\mathbf{k}}}} e^{-\omega_{\mathbf{k}}\tau}. \quad (17)$$

- Calculate its frequency components $b_{\mathbf{k}}$ and determine W according to eq. (15).
- The functional W should then be extremized over variables $b_{\mathbf{k}}$, $b_{\mathbf{k}}^*$ (or, what is the same, over all singularity surfaces of the described type), T and θ . The tree cross section of the process $1 \rightarrow n$ is then given by the formula (14).

Analytical solutions of this boundary value problem can be found only in special cases (they will be briefly described in section 5). Furthermore, in numerical computation it is impossible to extremize the functional (15) over infinite dimensional space of singularity surfaces. One can only make use of Rayleigh–Ritz procedure, i. e., choose some finite dimensional subclass of these surfaces and extremize the functional within this subclass. Let us consider this process more closely. Let the functional $\int d^d k b_{\mathbf{k}} b_{\mathbf{k}}^* e^{\omega_{\mathbf{k}} T}$ reach its minimal value for some $b_{\mathbf{k}}$: $\int d^d k b_{\mathbf{k}} b_{\mathbf{k}}^* e^{\omega_{\mathbf{k}} T} \big|_{\min} = C(T) > 0$. Let us fix some family of singularity surfaces $\Sigma(T)$. For these surfaces we have

¹Let us emphasize that we require only that τ_s *approaches* zero ($\tau_s \rightarrow 0$ at the point $\mathbf{x} = 0$).

$\int d^d k \, b_{\mathbf{k}} b_{\mathbf{k}}^* e^{\omega_{\mathbf{k}} T} \Big|_{\Sigma(T)} = C_{\Sigma}(T) \geq C(T)$ for all T . After inserting the saddle point value for θ , equal to $\theta = -\ln n + \ln C(T)$, we get

$$\begin{aligned} W(T) &= n \ln n - n + ET - n \ln C(T) && \text{has an extremum at } T_1, \\ W_{\Sigma}(T) &= n \ln n - n + ET - n \ln C_{\Sigma}(T) && \text{has an extremum at } T_2. \end{aligned}$$

Comparing $W(T_1)$ and $W_{\Sigma}(T_2)$ one can obtain the following inequalities:

$$\begin{aligned} W(T_1) &\geq W(T_2) \geq W_{\Sigma}(T_2) && \text{if } W \text{ has a maximum at } T_1; \\ W_{\Sigma}(T_2) &\leq W_{\Sigma}(T_1) \leq W(T_1) && \text{if } W_{\Sigma} \text{ has a minimum at } T_2 \end{aligned}$$

(in the real computation the second case is realized). Thus if we limit ourselves to some subclass of singularity surfaces, we get a lower bound on the exact value of $W_{\text{tree}}(E, n)$.

3 Expansion in spherical modes

The following calculations will be done in $(3+1)$ -dimensional space-time. We will consider only compact singularity surfaces.

The only requirement imposed on the singularity surface is that it touches the plane $\tau = 0$ at the point $\mathbf{x} = 0$ and $\tau_s(\mathbf{x}) < 0$ at all other spatial points. So, we can describe the singularity surface using the following method. Let us choose a sphere of the radius R_s with the centre at the origin of the coordinate system. Let the field configuration φ be infinite at the point $\tau = R_s$, $\mathbf{x} = 0$ and finite at all points $\sqrt{\mathbf{x}^2 + \tau^2} > R_s$. Then the singularity surface for this field touches the plane $\tau = R_s$ at $\mathbf{x} = 0$ and is contained inside the chosen sphere, i. e. at $\tau_s(\mathbf{x}) \leq R_s$. This description is suitable for singularity surfaces of the form of a sphere that is slightly squeezed along the horizontal direction. These configurations are ones of primary interest, as we will see from the results of the calculations. We should only make the substitution $\tau \rightarrow \tau + R_s$, to move the singularity to the origin. This is equivalent to the following change in frequency components of the field,

$$b_{\mathbf{k}} = \tilde{b}_{\mathbf{k}} e^{-\omega_{\mathbf{k}} R_s}, \quad (18)$$

where $\tilde{b}_{\mathbf{k}}$ are Fourier components of the field singular at the point $(R_s, 0)$.

As far as the field configuration is $O(3)$ symmetric, the field is a function of two variables, $\varphi(\rho, \theta)$, where θ is the angle between the radius-vector

and τ axis, and ρ is the length of the radius-vector (in 4-dimensional Euclidean space). The Euclidean field equations can be obtained by varying the following action:

$$S = 4\pi \int_0^\pi d\theta \int_{\rho_{\min}(\theta)}^\infty d\rho \rho^3 \sin^2 \theta \left\{ \frac{1}{2} \left(\frac{\partial \varphi}{\partial \rho} \right)^2 + \frac{1}{2\rho^2} \left(\frac{\partial \varphi}{\partial \theta} \right)^2 + \frac{1}{2} \varphi^2 + \frac{\lambda}{4} \varphi^4 \right\} . \quad (19)$$

Let us make use of the expansion in spherical modes,

$$\varphi(\rho, \theta) = \sum_{n=0}^{\infty} \varphi_n(\rho) C_n^{(1)}(\cos \theta) , \quad (20)$$

where $C_n^{(1)}(\cos \theta) = \frac{\sin(n+1)\theta}{\sin \theta}$ are Gegenbauer polynomials. Asymptotically, as $\rho \rightarrow \infty$, the functions $\varphi_n(\rho)$ are of the form

$$\varphi_n(\rho) = a_n \frac{K_{n+1}(\rho)}{\rho} , \quad (21)$$

where K_n are modified Bessel functions. The coefficients $\tilde{b}_{\mathbf{k}}$ of the expansion of this field configuration in plain waves are

$$\tilde{b}_{\mathbf{k}} = \sum_{n=0}^{\infty} a_n \sqrt{2\pi} \frac{C_n^{(1)}(\omega_{\mathbf{k}})}{\sqrt{2\omega_{\mathbf{k}}}} ,$$

so the integral in eq. (15) can be expressed in terms of the coefficients a_n as follows,

$$I(z) \equiv \int \tilde{b}_{\mathbf{k}}^* \tilde{b}_{\mathbf{k}} e^{-\omega_{\mathbf{k}} z} d^3 \mathbf{k} = 2\pi^2 \sum_{n,m=0}^{\infty} a_n a_m \left[K_{n+m+2}(z) - K_{n-m}(z) \right] \quad (22)$$

Upon substituting the mode expansion (20) into eq. (19), we get the expression for the action in terms of the spherical modes. Its extremization yields the equation (16) in terms of radial functions $\varphi_n(\rho)$.

The radial functions should have the form (21) at $\rho \rightarrow \infty$, i. e., they should not have growing components.

One also has to impose the second boundary condition which will ensure that the field becomes infinite on some singularity surface, subject to all

requirements mentioned in the beginning of this section. To formulate it precisely, one has to move a bit away from the singularity, i. e., the condition $\varphi(R_s, 0) = \infty$ should be substituted by the condition $\varphi(R, 0) = A$, where $A \gg 1/\sqrt{\lambda}$. In this case one can neglect the mass term in the field equation near the point $(R_s, 0)$ and approximate the singularity surface by a plane. Then φ in this region is

$$\varphi = \sqrt{\frac{2}{\lambda}} \frac{1}{l(x)} ,$$

where $l(x)$ is the distance from the point x to the singularity surface. It is straightforward to see that the singularity is placed at the following distance from the origin,

$$R_s = R - \sqrt{\frac{2}{\lambda}} \frac{1}{A} . \quad (23)$$

Thus, the singularity surface satisfying the required constraints (its form will be described in more detail later) is determined by a set of spherical components $c_n = \varphi_n(R)$, which should satisfy the condition

$$\sum_{n=0}^{\infty} c_n(n+1) = A , \quad (24)$$

i. e., $\varphi(R, \theta = 0) = A$; and also the condition

$$\varphi(R, \theta \neq 0) \leq A , \quad (25)$$

which in the simplest case of two non-zero components c_n is reduced to the requirement that both are non-negative.

The simplest configurations are $O(4)$ symmetric. They are defined in such a way that $c_0 = A$ and $c_n = 0$ for all other n , and are characterized by only one parameter, the radius of the singularity surface R .

Let us extremize over the parameters T , θ and singularity surfaces. Making use of eqs. (18) and (22), one writes the expression (15) in the following form,

$$W_{\text{tree}}(T, \theta, b_{\mathbf{k}}, b_{\mathbf{k}}^*) = ET - n\theta - I(z)e^{-\theta} ,$$

where $z = 2R_s - T$. The stationarity conditions for this expression over T and θ can be easily obtained,

$$n = I(z)e^{-\theta} , \quad E = -I'(z)e^{-\theta} , \quad (26)$$

where $I'(z)$ is the derivative of the expression (22). Finally,

$$W_{\text{tree}}(b_{\mathbf{k}}, b_{\mathbf{k}}^*) = n \ln \frac{\lambda n}{16} - n + n f(\varepsilon) , \quad (27)$$

$$f(\varepsilon) = (\varepsilon + 1)T + \ln 16 - \ln \lambda I(z) , \quad (28)$$

where T should be expressed through ε by solving the following equation,

$$\varepsilon + 1 = -\frac{I'(z)}{I(z)} , \quad (29)$$

which is a consequence of eq. (26).

For calculational reasons it is more convenient to perform the extremization in a slightly different order: first fix some value of T , then find the minimum of $I(z)$ over all singularity surfaces ($b_{\mathbf{k}}$ and $b_{\mathbf{k}}^*$) and, finally, obtain the corresponding value of ε from eq. (29).

Let us note that the function $f(\varepsilon)$ is independent of λ . Indeed, let us make the substitution

$$\varphi \rightarrow \frac{\tilde{\varphi}}{\sqrt{\lambda}} .$$

Then eq. (16) becomes the equation for $\tilde{\varphi}$ with $\lambda = 1$, and the integral (22) transforms into

$$I(z) \rightarrow \frac{\tilde{I}(z)}{\lambda} .$$

The dependence on λ in eq. (28) disappears. Thus one can set λ equal to one for the calculation of $f(\varepsilon)$.

In the case when $c_n \ll c_0$ (or, equivalently, $c_n \ll A$) for all $n > 0$, it is possible to determine the deviation of the singularity surface of the corresponding field configuration from sphere. In this case the field φ is large at $\rho = R$ for all θ , so in this region we can use the approximation of massless field. We will also assume that the radius R is large enough to consider the singularity surface flat at all points. Then, by making use of eq. (23), we immediately get

$$\Delta R_s(\theta) = \sqrt{\frac{2}{\lambda}} \left(\frac{1}{\varphi(R, \theta)} - \frac{1}{\varphi(R, 0)} \right) ,$$

where $\Delta R_s(\theta) = R_s(0) - R_s(\theta)$ characterizes the deviation of the singularity surface from sphere. The shape of a typical singularity surface found from our calculations is shown in fig. 1.

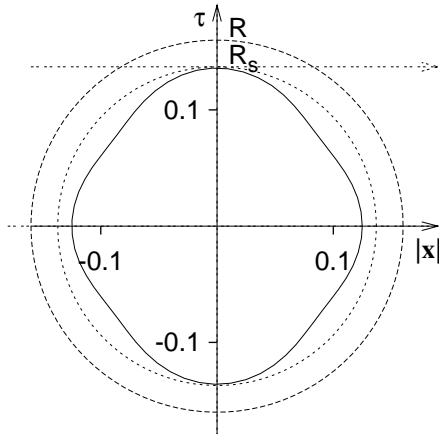


Figure 1: Singularity surface at energy $\varepsilon = 10$ (solid line). The 4th harmonic is turned on at infinity. A sphere of the radius R , on which one can determine c_n , is also shown.

4 Numerical calculation of tree cross sections

In the case of $O(4)$ -symmetric solutions (one-parameter family of singularity surfaces) the problem is simple: it is reduced to one ordinary differential equation for $\varphi_0(\rho)$. It is even not necessary to solve the boundary value problem, one can merely fix different values of a_0 (i. e., initial conditions at infinite ρ) and find the corresponding singularity radii R_s . The lower bound on $f(\varepsilon)$ obtained in this way is shown in fig. 2 by a solid line.

If we do not limit ourselves to spherically symmetric modes only, we should, in principle, solve the boundary value problem (16). It is difficult to solve it directly because one has to find φ_n that rapidly decrease with the harmonic number n to have the sum in eq. (22) convergent (and this is required at large ρ , where the field itself is small). Furthermore, in direct approach the configuration is determined by the shape of the singularity surface, where the values of the field are large, so the problem is extremely unstable. For the same reason one is unable to solve the boundary value problem formulated in terms of spherical components by fixing φ_n at the

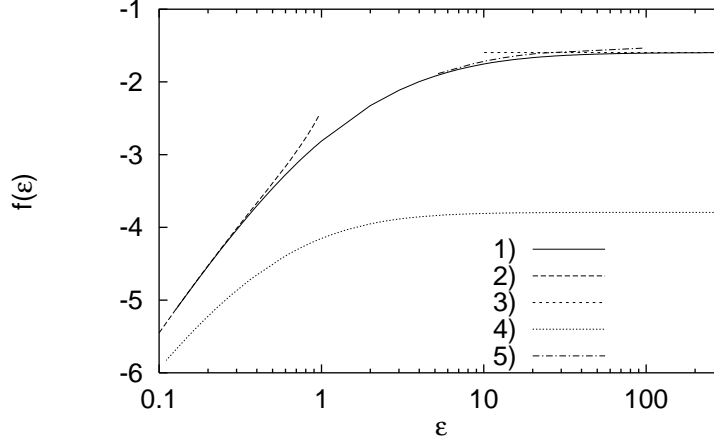


Figure 2: Bounds on $f(\varepsilon)$. 1) $O(4)$ -symmetric bound. 2) Analytical estimate for small energies ε . 3) Analytical lower bound for high energies. 4) Voloshin's lower bound. 5) The bound obtained with the 4th harmonic turned on at infinity.

radius R near the singularity.

However, one can invent a method that is similar to one used for the $O(4)$ -symmetric problem. Let us fix the values of a_n , i. e., spherical modes at infinity (that is equivalent to fixing $b_{\mathbf{k}}$) and solve the system of *ordinary* differential equations with *initial* conditions specified in this way. At some finite radius R the field becomes infinite (more precisely, becomes larger than some prescribed large number A). If the coefficients $c_n = \varphi_n(R)$ at this R obey the conditions (24) and (25), then this configuration satisfies all requirements mentioned above. We found that this property is satisfied in a sufficiently large region of parameters a_n , i. e., radial functions at infinity. In particular, one can set all a_n equal to zero for all $n \neq 0, k$ for any fixed k .

The major problem of this method (and presumably any method that makes use of the expansion in frequency components) is that the number of spherical modes used for computation is limited. However, the errors caused by this truncation are significant only at short distances from the singularity (in the region of strong non-linearity) and do not lead to a considerable error

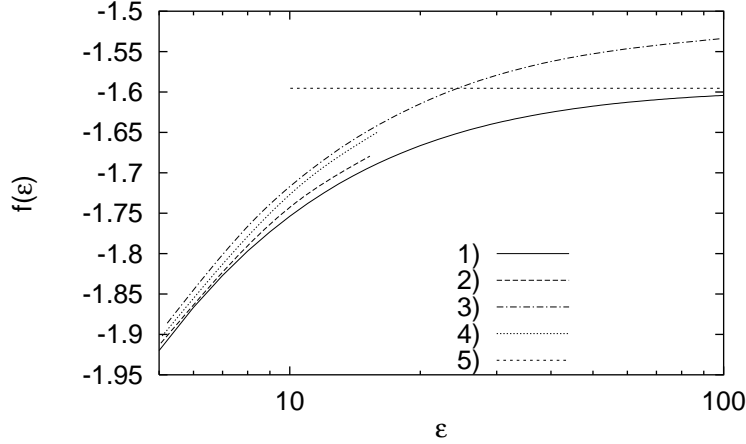


Figure 3: Bounds on $f(\varepsilon)$ (scaled part of fig. 2). 1) $O(4)$ -symmetric bound. 2) Bound obtained with the 2nd harmonic turned on at infinity. 3) 4th harmonic. 4) 6th harmonic. 5) Analytical lower bound for high energies.

in the determination of R_s , at least for not too high energies.

To search for the minimum of the $I(2R_s - T)$ as a function of a_n (that effectively means the minimization over $b_{\mathbf{k}}, b_{\mathbf{k}}^*$) we used the multidimensional downhill simplex method [11], that does not require any additional information about the function which is extremized (i. e. it does not need its derivatives). To find the minimum, about 50 calculations of $I(2R_s - T)$ had to be made for different configurations.

The calculations were performed with turning on two non-zero spherical modes at infinity, a_0 and a_k . All values of k smaller than 8 were explored. The maximum deviation from the $O(4)$ -symmetric result was obtained at $k = 4$. Let us recall in this respect that we obtain the lower bound on $f(\varepsilon)$ by the Rayleigh–Ritz procedure, so we are interested in the maximum value of $f(\varepsilon)$. The results for $k = 4, 2$ and 6 are shown in the fig. 3. At smaller energies ε the difference from the spherical calculation is negligible while at larger ε the singularity radius R_s becomes small and relative error in its calculation grows. The $O(4)$ -symmetric calculations were performed in a wider interval of energies, which is shown in fig. 2 together with some

analytical results (see section 5).

The shape of the singularity surface corresponding to the minimal value of $I(2R_s - T)$ is also of some interest. The singularity surface for $\varepsilon = 10$ and $k = 4$ is shown in fig. 1.

5 Comparison of analytical and numerical results

In the limiting cases of small and large energies it is possible to implement the procedure of the section 2 analytically [4, 12]. Let us present the analytical results here for comparison.

In case of small energy ε one may start with the solutions of the following form,

$$\varphi(\tau, \mathbf{x}) = \sqrt{\frac{2}{\lambda}} \frac{1}{\sinh(\tau - \tau_s(\mathbf{x}))} ,$$

which satisfies eq. (16) up to the terms of order $O((\partial_{\mathbf{x}}\tau_s)^2)$ (the case when $\tau_s(\mathbf{x}) = 0$ for all \mathbf{x} corresponds to the creation of particles at the threshold). One can also find corrections of the orders $(\partial_{\mathbf{x}}\tau_s)^2$ and $(\partial_{\mathbf{x}}\tau_s)^4$ to this expression. As a result, one obtains the following estimate [4, 12],

$$f(\varepsilon) = \frac{3}{2} \left(\ln \frac{\varepsilon}{3\pi} + 1 \right) - \varepsilon \frac{17}{12} + \frac{\varepsilon^2}{432} (1327 - 96\pi^2) + O(\varepsilon^3) . \quad (30)$$

This analytical result is shown in fig. 2 by a dashed line; it coincides with our numerical result at $\varepsilon < 0.5$.

In the case of high energies one may try to neglect the mass term in the field equation and consider the massless φ^4 theory. In massless theory, an $O(4)$ -symmetric solution—the Fubini–Lipatov instanton—is known [13, 14]. This solution may be used to construct the solution which is singular at the point $\tau = \mathbf{x} = 0$ [15]:

$$\varphi_0 = \sqrt{\frac{8}{\lambda}} \frac{R_s}{\mathbf{x}^2 + (\tau + R_s)^2 - R_s^2} , \quad (31)$$

where R_s is the collective coordinate determining the size of the singularity surface. From this solution one obtains the following bound [4, 15],

$$f(\varepsilon \rightarrow \infty) \geq \ln \frac{2}{\pi^2} . \quad (32)$$

This bound is consistent with our numerical calculations at $\varepsilon > 50$.

It is of interest to compare our numerical results with other existing estimates. An alternative lower bound on $f(\varepsilon)$, following from the direct analysis of Feynman diagrams, is given in ref. [16]. As is clear from fig. 2 our bound is considerably more stringent.

6 Conclusion

In this paper we described a method for the calculation of multiparticle cross sections in φ^4 theory at the tree level. Our numerical results obtained in the $O(4)$ -symmetric case coincide with known limiting cases in the domains of validity of the latter. More precise results obtained with larger family of singularity surfaces indicate that even the simplest $O(4)$ -symmetric approximation gives nearly an exact answer. The lower bounds obtained in this paper are better than previous bounds.

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